Generalized sampling: From shift-invariant to U-invariant spaces

H. R. Fernández-Morales and A. G. García

M. A. Hernández-Medina

M. J. Muñoz-Bouzo

The aim of this article is to derive a sampling theory in U-invariant subspaces of a separable Hilbert space $\mathcal{H}$ where U denotes a unitary operator defined on $\mathcal{H}$. To this end, we use some special dual frames for $L^2(0,1)$, and the fact that any U-invariant subspace with stable generator is the image of $L^2(0,1)$ by means of a bounded invertible operator. The used mathematical technique mimics some previous sampling work for shift-invariant subspaces of $L^2(\mathbb{R})$. Thus, sampling frame expansions in U-invariant spaces are obtained. In order to generalize convolution systems and deal with the time-jitter error in this new setting we consider a continuous group of unitary operators which includes the operator U.

Keywords: Stationary sequences; U-invariant subspaces; frames; dual frames; time-jitter error; group of unitary operators; pseudo-dual frames.

1. By Way of Motivation

The aim in this paper is to derive a generalized sampling theory for U-invariant subspaces of a separable Hilbert space $\mathcal{H}$, where $U : \mathcal{H} \to \mathcal{H}$ denotes a unitary operator. The motivation for our work can be found in the generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$; there $\mathcal{H} := L^2(\mathbb{R})$ and the unitary operator is the shift $T: f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$. In that setting, the functions (signals) belong to some (principal) shift-invariant subspace $V_\varphi^2 := \overline{\operatorname{span}}_{L^2(\mathbb{R})}\{\varphi(u-n),\ n \in \mathbb{Z}\}$, where the generator function φ belongs to $L^2(\mathbb{R})$ and the sequence $\{\varphi(u-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$. Thus, the shift-invariant space V_φ^2 can be described as

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \varphi(u-n) : \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}.$$

On the other hand, in many common situations the available data are samples of some filtered versions $f * \mathsf{h}_j$ of the signal f itself, where the average function h_j reflects the characteristics of the adquisition device.

For s convolution systems (linear time-invariant systems or filters in engineering jargon) $\mathcal{L}_j f := f * \mathsf{h}_j$, $j = 1, 2, \dots, s$, defined on V_φ^2, and assuming also that the sequence of samples

$$\{(\mathcal{L}_j f)(rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s},$$

where $r \in \mathbb{N}$, is available for any f in V_φ^2, the generalized sampling problem mathematically consists of the stable recovery of any $f \in V_\varphi^2$ from the above sequence of samples. In other words, it deals with the construction of sampling formulas in V_φ^2 having the form

$$f(u) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} (\mathcal{L}_j f)(rm) S_j(u - rm), \quad u \in \mathbb{R},$$

where the sequence of reconstruction functions $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for the shift-invariant space V_φ^2.

Sampling in shift-invariant spaces of $L^2(\mathbb{R})$ (or $L^2(\mathbb{R}^d)$), with one or multiple generators, has been profusely treated in the mathematical literature. A few selected references are: [4, 5, 9–14, 18, 23, 27–31].

In this work we provide a generalization of the above problem in the following sense. Let U be a unitary operator in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by $\mathcal{A}_a := \overline{\operatorname{span}}\{U^n a, n \in \mathbb{Z}\}$. In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n U^n a : \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}.$$

In order to generalize convolution systems and mainly to obtain some perturbation results in this new setting, we assume that the operator U is included in a continuous *group of unitary operators* $\{U^t\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ as $U := U^1$. Recall that $\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying (see [2, Vol. 2; p. 29]):

(1) $U^t U^{t'} = U^{t+t'}$,

(2) $U^0 = I_{\mathcal{H}}$,
(3) $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

Note that $(U^t)^{-1} = U^{-t}$, and since $(U^t)^* = (U^t)^{-1}$, we have $(U^t)^* = U^{-t}$.

Thus, for $b \in \mathcal{H}$ we consider the linear operator $\mathcal{H} \ni x \mapsto \mathcal{L}_b x \in C(\mathbb{R})$ such that $(\mathcal{L}_b x)(t) := \langle x, U^t b\rangle_{\mathcal{H}}$ for every $t \in \mathbb{R}$. These operators $\mathcal{L}_b$, which will be called U-systems, can be seen as a generalization of the convolution systems in $L^2(\mathbb{R})$. Indeed, for the shift operator $T : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$ we have

$$\langle f, T^t b\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(u)\overline{b(u-t)}du = (f * h)(t), \quad t \in \mathbb{R},$$

where $h(u) := \overline{b(-u)}$.

Given U-systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, corresponding to s elements $b_j \in \mathcal{H}$, i.e. $\mathcal{L}_j \equiv \mathcal{L}_{b_j}$ for each $j = 1, 2, \ldots, s$, the generalized regular sampling problem in $\mathcal{A}_a$ consists of the stable recovery of any $x \in \mathcal{A}_a$ from the sequence of the samples

$$\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z}; j=1,2,\ldots,s} \quad \text{where } r \in \mathbb{N}, \ \ r \geq 1.$$

This U-sampling problem has been treated, for the first time, in some recent papers [22, 24]. Sampling in shift-invariant subspaces or in modulation-invariant subspaces of $L^2(\mathbb{R})$ becomes a particular case of U-sampling associated with the shift operator $T : f(u) \mapsto f(u-1)$ or with the modulation operator $M : f(u) \mapsto \mathrm{e}^{2\pi \mathrm{i} u} f(u)$ in $L^2(\mathbb{R})$ respectively.

In this paper, we propose a completely different approach which allows to analyze in depth the U-sampling problem. In Sec. 3, we prove the existence of frames in $\mathcal{A}_a$, having the form $\{U^{rm} c_j\}_{m\in\mathbb{Z}; j=1,2,\ldots,s}$, where $c_j \in \mathcal{A}_a$ for $j = 1, 2, \ldots, s$, such that for each $x \in \mathcal{A}_a$ the sampling expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm) U^{rm} c_j \quad \text{in } \mathcal{H} \tag{1.1}$$

holds. To this end, as in the shift-invariant case (see, for instance, [13, 14]), we use that the above sampling formula is intimately related with some special dual frames in $L^2(0,1)$ (see Sec. 2) via the isomorphism $\mathcal{T}_{U,a} : L^2(0,1) \to \mathcal{A}_a$ which maps the orthonormal basis $\{\mathrm{e}^{2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$. In [24], regular sampling expansions like (1.1) are obtained by using a completely different technique; basically, they use the cross-covariance function $R_{a,b_j}(n) := \langle U^n a, b_j\rangle_{\mathcal{H}}$ between the sequences $\{U^n a\}_{n\in\mathbb{Z}}$ and $\{U^n b_j\}_{n\in\mathbb{Z}}$, $j = 1, 2, \ldots, s$.

Strictly speaking, we do not need the formalism of the continuous group of unitary operators to derive the sampling results in Sec. 3 since we only use the discrete group $\{U^n\}_{n\in\mathbb{Z}}$ completely determined by U. However, for the study, in Sec. 4, of the time-jitter error in sampling formulas as in (1.1), the continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ becomes essential. In this case, we dispose of a perturbed sequence of samples $\{(\mathcal{L}_j x)(rm + \epsilon_{mj})\}_{m\in\mathbb{Z}; j=1,2,\ldots,s}$, with errors $\epsilon_{mj} \in \mathbb{R}$, for the recovery of $x \in \mathcal{A}_a$. We prove that, for small enough errors ϵ_{mj}, the stable recovery of any $x \in \mathcal{A}_a$ is still possible. Finally, in Sec. 5 we deal with the case of multiple stable generators. We only sketch the procedure since it is essentially identical to the one-generator case.

2. On Sampling in U-Invariant Subspaces

For a fixed $a \in \mathcal{H}$, assume that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$. Recall that a *Riesz basis* in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n\in\mathbb{Z}}$ has a unique biorthogonal (dual) Riesz basis $\{y_n\}_{n\in\mathbb{Z}}$, i.e. $\langle x_n, y_m\rangle_{\mathcal{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n\in\mathbb{Z}} \langle x, y_n\rangle_{\mathcal{H}} x_n = \sum_{n\in\mathbb{Z}} \langle x, x_n\rangle_{\mathcal{H}} y_n,$$

hold for every $x \in \mathcal{H}$. We state the definition by considering the integers set $\mathbb{Z}$ as the index set since throughout the paper most of sequences are indexed in $\mathbb{Z}$. A *Riesz sequence* in $\mathcal{H}$ is a Riesz basis for its closed span (see, for instance, [8]). Thus, the U-invariant subspace $\mathcal{A}_a := \overline{\operatorname{span}}\{U^n a, n \in \mathbb{Z}\}$ can be expressed as

$$\mathcal{A}_a = \left\{ \sum_{n\in\mathbb{Z}} \alpha_n U^n a : \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \right\}.$$

For simplicity and ease of notation we are considering the one-generator setting; as we have already said, the same sampling results for the general case can be obtained by analogy, and it will be drawn in Sec. 5. The sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a *stationary sequence* since the inner product $\langle U^n a, U^m a\rangle_{\mathcal{H}}$ depends only on the difference $n - m \in \mathbb{Z}$. Moreover, the *auto-covariance* R_a of the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta), \quad k \in \mathbb{Z},$$

in terms of a positive Borel measure μ_a on $(-\pi, \pi)$ called the *spectral measure* of the sequence (see [19]). This is obtained from the integral representation of the unitary operator U on $\mathcal{H}$ (see, for instance, [2, 33]). The spectral measure μ_a can be decomposed into an absolute continuous and a singular part as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$. A necessary and sufficient condition in order for the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ to be a Riesz sequence for $\mathcal{H}$ is given in next theorem in terms of the decomposition of the spectral measure μ_a.

Theorem 2.1. *Let $\{U^n a\}_{n\in\mathbb{Z}}$ be a sequence obtained from a unitary operator in a separable Hilbert space $\mathcal{H}$ with spectral measure $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$, and let $\mathcal{A}_a$ be the closed subspace spanned by $\{U^n a\}_{n\in\mathbb{Z}}$. Then the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s \equiv 0$ and*

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \leq \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty.$$

Theorem 2.1 is just the one-generator case ($L = 1$) of Theorem 5.1 proved below. It is worth to mention that a straightforward computation shows that the dual Riesz basis of $\{U^n a\}_{n\in\mathbb{Z}}$ in $\mathcal{A}_a$ is given by $\{U^n b\}_{n\in\mathbb{Z}}$ with $b = \sum_{k\in\mathbb{Z}} b_k U^k a \in \mathcal{A}_a$, where the terms of the sequence $\{b_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ are the Fourier coefficients of the function $1/\phi_a(\theta) \in L^2(-\pi,\pi)$. Indeed, for $b = \sum_{k\in\mathbb{Z}} b_k U^k a$ in $\mathcal{A}_a$, the biorthogonality between the sequences $\{U^n a\}_{n\in\mathbb{Z}}$ and $\{U^n b\}_{n\in\mathbb{Z}}$ means

$$\begin{aligned}
\delta_{m,0} = \langle U^m a, b\rangle_{\mathcal{H}} = \left\langle U^m a, \sum_{k\in\mathbb{Z}} b_k U^k a\right\rangle_{\mathcal{H}} &= \sum_{k\in\mathbb{Z}} \overline{b}_k \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i(m-k)\theta}\phi_a(\theta)d\theta \\
= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k\in\mathbb{Z}} \overline{b}_k \mathrm{e}^{-ik\theta}\right)\phi_a(\theta)\mathrm{e}^{im\theta}d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\theta)\phi_a(\theta)\mathrm{e}^{-im\theta}d\theta,
\end{aligned}$$

where $B(\theta) := \sum_{k\in\mathbb{Z}} b_k \mathrm{e}^{ik\theta}$; in other words, we have $B(\theta)\phi_a(\theta) \equiv 1$ in $L^2(-\pi,\pi)$. Moreover, it is easy to deduce that $\phi_b(\theta) = 1/\phi_a(\theta)$, $\theta \in (-\pi,\pi)$; that is, for $k \in \mathbb{Z}$ we obtain $\langle U^k b, b\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{ik\theta}\frac{d\theta}{\phi_a(\theta)}$.

Finally, for the shift operator $T : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$, Theorem 2.1 allows to recover the classical necessary and sufficient condition for the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$, where $\varphi \in L^2(\mathbb{R})$, to be a Riesz basis for the corresponding shift-invariant subspace $\mathcal{A}_\varphi$ in $L^2(\mathbb{R})$. Indeed, consider the Fourier transform as $\widehat{\varphi}(\theta) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(t)\mathrm{e}^{-it\theta}d\theta$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$; using the Parseval's equality one easily gets

$$\begin{aligned}
\langle T^k\varphi, \varphi\rangle_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} \varphi(u-k)\overline{\varphi(u)}du = \int_{-\infty}^{\infty} \widehat{\varphi(u-k)}(\theta)\overline{\widehat{\varphi}(\theta)}d\theta \\
&= \int_{-\infty}^{\infty} |\widehat{\varphi}(\theta)|^2 \mathrm{e}^{-ik\theta}d\theta = \int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2\mathrm{e}^{-ik\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-ik\theta}2\pi\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(-\theta+2\pi n)|^2 d\theta,
\end{aligned}$$

that is, $\phi_\varphi(\theta) = 2\pi\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(-\theta+2\pi n)|^2$, $\theta \in (-\pi,\pi)$. Thus, Theorem 2.1 yields the classical condition (see, for instance, [8]):

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2 \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2 < \infty.$$

The following isomorphism between $L^2(0,1)$ and $\mathcal{A}_a$ will be crucial along this paper.

The isomorphism $\mathcal{T}_{U,a}$

We define the isomorphism $\mathcal{T}_{U,a}$ which maps the orthonormal basis $\{\mathrm{e}^{2\pi inw}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$, that is,

$$\begin{aligned}
\mathcal{T}_{U,a} &: L^2(0,1) \to \mathcal{A}_a, \\
F = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{2\pi inw} &\mapsto x = \sum_{n\in\mathbb{Z}} \alpha_n U^n a.
\end{aligned}$$

The following U-shift property holds: For any $F \in L^2(0,1)$ and $N \in \mathbb{Z}$, we have

$$\mathcal{T}_{U,a}(F\,\mathrm{e}^{2\pi\mathrm{i}Nw}) = U^N(\mathcal{T}_{U,a}F). \tag{2.1}$$

The U-systems

For any fixed $b \in \mathcal{H}$ we define the U-system $\mathcal{L}_b$ as the linear operator between $\mathcal{H}$ and the set $C(\mathbb{R})$ of the continuous functions on $\mathbb{R}$ given by

$$\mathcal{H} \ni x \mapsto \mathcal{L}_b x \in C(\mathbb{R}) \quad \text{such that } \mathcal{L}_b x(t) := \langle x, U^t b\rangle_{\mathcal{H}},\ \ t \in \mathbb{R}.$$

For any $x \in \mathcal{A}_a$ and $t \in \mathbb{R}$, by using the Plancherel equality for the orthonormal basis $\{\mathrm{e}^{2\pi inw}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$, we have

$$\begin{aligned}\mathcal{L}_b x(t) = \langle x, U^t b\rangle_{\mathcal{H}} &= \Big\langle \sum_{n\in\mathbb{Z}} \alpha_n U^n a, U^t b\Big\rangle_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} \alpha_n \overline{\langle U^t b, U^n a\rangle}_{\mathcal{H}} \\ &= \Big\langle F, \sum_{n\in\mathbb{Z}} \langle U^t b, U^n a\rangle_{\mathcal{H}} \mathrm{e}^{2\pi inw}\Big\rangle_{L^2(0,1)} = \langle F, K_t\rangle_{L^2(0,1)},\end{aligned} \tag{2.2}$$

where $\mathcal{T}_{U,a}F = x$, and the function

$$K_t(w) := \sum_{n\in\mathbb{Z}} \langle U^t b, U^n a\rangle_{\mathcal{H}} \mathrm{e}^{2\pi inw} = \sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_b a(t-n)} \mathrm{e}^{2\pi inw}$$

belongs to $L^2(0,1)$ since the sequence $\{\langle U^t b, U^n a\rangle_{\mathcal{H}}\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ for each $t \in \mathbb{R}$.

An expression for the generalized samples

Suppose that s vectors $b_j \in \mathcal{H}$, $j = 1, 2, \ldots, s$, are given and consider their associated U-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$, $j = 1, 2, \ldots, s$. Our aim is the stable recovery of any $x \in \mathcal{A}_a$ from the sequence of samples $\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};j=1,2,\ldots,s}$ where $r \geq 1$. To this end, first we obtain a suitable expression for the samples. For $x \in \mathcal{A}_a$ let $F \in L^2(0,1)$ such that $\mathcal{T}_{U,a}F = x$; by using (2.2), for $j = 1, 2, \ldots, s$ and $m \in \mathbb{Z}$ we have

$$\begin{aligned}\mathcal{L}_j x(rm) &= \Big\langle F, \sum_{n\in\mathbb{Z}} \langle U^{rm} b_j, U^n a\rangle_{\mathcal{H}} \mathrm{e}^{2\pi inw}\Big\rangle_{L^2(0,1)} \\ &= \Big\langle F, \sum_{k\in\mathbb{Z}} \langle U^k b_j, a\rangle_{\mathcal{H}} \mathrm{e}^{2\pi\mathrm{i}(rm-k)w}\Big\rangle_{L^2(0,1)} \\ &= \Big\langle F, \Big[\sum_{k\in\mathbb{Z}} \overline{\langle a, U^k b_j\rangle}_{\mathcal{H}} \mathrm{e}^{-2\pi\mathrm{i}kw}\Big] \mathrm{e}^{2\pi\mathrm{i}rmw}\Big\rangle_{L^2(0,1)},\end{aligned}$$

where the change in the summation's index $k := rm - n$ has been done. Hence,

$$\mathcal{L}_j x(rm) = \langle F, \overline{g_j(w)} \mathrm{e}^{2\pi\mathrm{i}rmw}\rangle_{L^2(0,1)} \quad \text{for } m \in \mathbb{Z} \quad \text{and} \quad j = 1, 2, \ldots, s, \tag{2.3}$$

where the function

$$g_j(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k) \mathrm{e}^{2\pi i k w} \tag{2.4}$$

belongs to $L^2(0,1)$ for each $j = 1, 2, \dots, s$.

As a consequence of (2.3), the stable recovery of any $x \in \mathcal{A}_a$ depends on whether the sequence $\{\overline{g_j(w)}\mathrm{e}^{2\pi i r m w}\}_{m\in\mathbb{Z}; j=1,2,\dots,s}$ forms a frame for $L^2(0,1)$. Recall that a sequence $\{x_n\}_{n\in\mathbb{Z}}$ is a *frame* for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|x\|^2 \le \sum_{n\in\mathbb{Z}} |\langle x, x_n\rangle|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ satisfying only the right-hand side inequality above is said to be a *Bessel sequence* for $\mathcal{H}$. Given a frame $\{x_n\}_{n\in\mathbb{Z}}$ for $\mathcal{H}$ the representation property of any vector $x \in \mathcal{H}$ as a series $x = \sum_{n\in\mathbb{Z}} c_n x_n$ is retained, but, unlike the case of Riesz bases (*exact frames*), the uniqueness of this representation (for *overcomplete frames*) is sacrificed. Suitable frame coefficients c_n which depend continuously and linearly on x are obtained by using the dual frames $\{y_n\}_{n\in\mathbb{Z}}$ of $\{x_n\}_{n\in\mathbb{Z}}$, i.e. $\{y_n\}_{n\in\mathbb{Z}}$ is another frame for $\mathcal{H}$ such that $x = \sum_{n\in\mathbb{Z}}\langle x, y_n\rangle x_n = \sum_{n\in\mathbb{Z}}\langle x, x_n\rangle y_n$ for each $x \in \mathcal{H}$. For more details on frame theory see [8].

A deep study of sequences having the form of $\{\overline{g_j(w)}\mathrm{e}^{2\pi i r m w}\}_{m\in\mathbb{Z}; j=1,2,\dots,s}$ was done in [13, 14]. Namely, consider the $s \times r$ matrix of functions in $L^2(0,1)$

$$\begin{aligned}
\mathbb{G}(w) &:= \begin{bmatrix}
g_1(w) & g_1\left(w + \frac{1}{r}\right) & \cdots & g_1\left(w + \frac{r-1}{r}\right) \\
g_2(w) & g_2\left(w + \frac{1}{r}\right) & \cdots & g_2\left(w + \frac{r-1}{r}\right) \\
\vdots & \vdots & & \vdots \\
g_s(w) & g_s\left(w + \frac{1}{r}\right) & \cdots & g_s\left(w + \frac{r-1}{r}\right)
\end{bmatrix} \\
&= \left[g_j\left(w + \frac{k-1}{r}\right)\right]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r}}
\end{aligned} \tag{2.5}$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)], \quad \beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

where $\mathbb{G}^*(w)$ denotes the transpose conjugate of the matrix $\mathbb{G}(w)$, and $\lambda_{\min}$ (respectively, $\lambda_{\max}$) the smallest (respectively, the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(w)\mathbb{G}(w)$. Observe that $0 \le \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} \le \infty$. Notice that in the definition of the matrix $\mathbb{G}(w)$ we are considering 1-periodic extensions of the involved functions g_j, $j = 1, 2, \dots, s$.

A complete characterization of the sequence $\{\overline{g_j(w)}\mathrm{e}^{2\pi\mathrm{i}rmw}\}_{m\in\mathbb{Z};j=1,2,\ldots,s}$ is given in the next lemma (see [14, Lemma 3] or [13, Lemma 2] for the proof).

Lemma 2.2. *For the functions $g_j \in L^2(0,1)$, $j = 1,2,\ldots,s$, consider the associated matrix $\mathbb{G}(w)$ given in (2.5). Then, the following results hold:*

(a) *The sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi\mathrm{i}rnw}\}_{n\in\mathbb{Z};j=1,2,\ldots,s}$ is a complete system for $L^2(0,1)$ if and only if the rank of the matrix $\mathbb{G}(w)$ is r a.e. in $(0,1/r)$.*
(b) *The sequence $\{\overline{g_j(w)}\mathrm{e}^{2\pi\mathrm{i}rnw}\}_{n\in\mathbb{Z};j=1,2,\ldots,s}$ is a Bessel sequence for $L^2(0,1)$ if and only if $g_j \in L^\infty(0,1)$ (or equivalently $\beta_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{G}}/r$.*
(c) *The sequence $\{\overline{g_j(w)}\mathrm{e}^{2\pi\mathrm{i}rnw}\}_{n\in\mathbb{Z};j=1,2,\ldots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.*
(d) *The sequence $\{\overline{g_j(w)}\mathrm{e}^{2\pi\mathrm{i}rnw}\}_{n\in\mathbb{Z};j=1,2,\ldots,s}$ is a Riesz basis for $L^2(0,1)$ if and only if is a frame and $s = r$.*

A comment about Lemma 2.2 in terms of the average sampling terminology introduced by Aldroubi *et al.* in [6] is in order. According to [6], we say that:

(1) The set $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s\}$ is an *r-determining U-sampler* for $\mathcal{A}_a$ if the only vector $x \in \mathcal{A}_a$, satisfying $\mathcal{L}_j x(rm) = 0$ for all $j = 1,2,\ldots,s$ and $m \in \mathbb{Z}$, is $x = 0$.
(2) The set $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s\}$ is an *r-stable U-sampler* for $\mathcal{A}_a$ if there exist positive constants A and B such that

$$A\|x\|^2 \leq \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} |\mathcal{L}_j x(rm)|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{A}_a.$$

Hence, parts (a) and (c) of Lemma 2.2 can be read, by using (2.3), as follows:

(i) The set $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s\}$ is an r-determining U-sampler for $\mathcal{A}_a$ if and only if $\operatorname{rank}\mathbb{G}(w) = r$ a.e. in $(0,1)$ (and hence, necessarily, $s \geq r$).
(ii) The set $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s\}$ is an r-stable U-sampler for $\mathcal{A}_a$ if and only if $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$.

An r-determining U-sampler for $\mathcal{A}_a$ can distinguish between two distinct elements in $\mathcal{A}_a$, but the recovery, if any, is not necessarily stable. If the system $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s\}$ is an r-stable U-sampler for $\mathcal{A}_a$, then any $x \in \mathcal{A}_a$ can be recovered, in a stable way, from the sequence of generalized samples $\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};j=1,2,\ldots,s}$, where necessarily $s \geq r$. Roughly speaking, the operator which maps

$$\mathcal{A}_a \ni x \mapsto \{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};j=1,2,\ldots,s} \in \ell^2_s(\mathbb{Z}) := \underbrace{\ell^2(\mathbb{Z}) \times \cdots \times \ell^2(\mathbb{Z})}_{(s \text{ times})}$$

has a bounded inverse.

Having in mind (2.3), from the sequence of samples $\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};j=1,2,\ldots,s}$ we recover $F \in L^2(0,1)$, and by means of the isomorphism $\mathcal{T}_{U,a}$, the vector $x = \mathcal{T}_{U,a}F$ in $\mathcal{A}_a$. This will be the main goal in the next section.

3. Generalized Regular Sampling in $\mathcal{A}_a$

Along with the characterization of the sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi i rnw}\}_{n\in\mathbb{Z};j=1,2,\dots,s}$ as a frame in $L^2(0,1)$, in [14] a family of dual frames are also given: Choose functions h_j in $L^\infty(0,1)$, $j=1,2,\dots,s$, such that

$$[h_1(w), h_2(w), \dots, h_s(w)]\mathbb{G}(w) = [1, 0, \dots, 0] \quad \text{a.e. in } (0,1). \tag{3.1}$$

It was proven in [14] that the sequence $\{rh_j(w)\mathrm{e}^{2\pi i rnw}\}_{n\in\mathbb{Z};j=1,2,\dots,s}$ is a dual frame of the sequence $\{\overline{g_j(w)}\mathrm{e}^{2\pi i rnw}\}_{n\in\mathbb{Z};j=1,2,\dots,s}$ in $L^2(0,1)$. In other words, taking into account (2.3), we have for any $F \in L^2(0,1)$ the expansion

$$F = \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm) r h_j(w)\mathrm{e}^{2\pi i rmw} \quad \text{in } L^2(0,1). \tag{3.2}$$

Concerning to the existence of the functions h_j, $j=1,2,\dots,s$, consider the first row of the $r\times s$ Moore–Penrose pseudo-inverse $\mathbb{G}^\dagger(w)$ of $\mathbb{G}(w)$ given by

$$\mathbb{G}^\dagger(w) := [\mathbb{G}^*(w)\mathbb{G}(w)]^{-1}\mathbb{G}^*(w).$$

Its entries are essentially bounded in $(0,1)$ since the functions g_j, $j=1,2,\dots,s$, and $\det^{-1}[\mathbb{G}^*(w)\mathbb{G}(w)]$ are essentially bounded in $(0,1)$, and (3.1) trivially holds. All the possible solutions of (3.1) are given by the first row of the $r\times s$ matrices given by

$$\mathbb{H}_{\mathbb{K}}(w) := \mathbb{G}^\dagger(w) + \mathbb{K}(w)[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^\dagger(w)], \tag{3.3}$$

where $\mathbb{K}(w)$ denotes any $r\times s$ matrix with entries in $L^\infty(0,1)$, and $\mathbb{I}_s$ is the identity matrix of order s.

Applying the isomorphism $\mathcal{T}_{U,a}$ in (3.2), for $x = \mathcal{T}_{U,a}F \in \mathcal{A}_a$ we obtain the sampling expansion:

$$\begin{aligned} x &= \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\mathcal{T}_{U,a}[rh_j(\cdot)\mathrm{e}^{2\pi i rm\cdot}] = \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)U^{rm}[\mathcal{T}_{U,a}(rh_j)] \\ &= \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)U^{rm}c_{j,h} \quad \text{in } \mathcal{H}, \end{aligned} \tag{3.4}$$

where $c_{j,h} := \mathcal{T}_{U,a}(rh_j) \in \mathcal{A}_a$, $j=1,2,\dots,s$, and we have used the U-shift property (2.1). Besides, the sequence $\{U^{rm}c_{j,h}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$. In fact, the following result holds.

Theorem 3.1. *Let $b_j \in \mathcal{H}$ and let $\mathcal{L}_j$ be its associated U-system for $j=1,2,\dots,s$. Assume that the function g_j, $j=1,2,\dots,s$, given in* (2.4) *belongs to $L^\infty(0,1)$; or equivalently, that $\beta_{\mathbb{G}} < \infty$ for the associated $s\times r$ matrix $\mathbb{G}(w)$. The following statements are equivalent:*

(a) $\alpha_{\mathbb{G}} > 0$.

(b) *There exists a vector* $[h_1(w), h_2(w), \dots, h_s(w)]$ *with entries in* $L^\infty(0,1)$ *satisfying*

$$[h_1(w), h_2(w), \dots, h_s(w)]\mathbb{G}(w) = [1, 0, \dots, 0] \quad \textit{a.e. in } (0,1).$$

(c) *There exist elements* $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, *such that the sequence* $\{U^{rk}c_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ *is a frame for* $\mathcal{A}_a$, *and for any* $x \in \mathcal{A}_a$ *the expansion*

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk) U^{rk} c_j \quad \textit{in } \mathcal{H}, \tag{3.5}$$

holds.

(d) *There exists a frame* $\{C_{j,k}\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ *for* $\mathcal{A}_a$ *such that, for each* $x \in \mathcal{A}_a$ *the expansion*

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk) C_{j,k} \quad \textit{in } \mathcal{H},$$

holds.

Proof. We have already proved that (a) implies (b) and that (b) implies (c). Obviously, (c) implies (d). As a consequence, we only need to prove that (d) implies (a). Applying the isomorphism $\mathcal{T}_{U,a}^{-1}$ to the expansion in (d), and taking into account (2.3) we obtain

$$\begin{aligned} F = \mathcal{T}_{U,a}^{-1} x &= \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk) \mathcal{T}_{U,a}^{-1}(C_{j,k}) \\ &= \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle F, \overline{g_j(w)} e^{2\pi i rkw} \rangle_{L^2(0,1)} \mathcal{T}_{U,a}^{-1}(C_{j,k}) \quad \text{in } L^2(0,1), \end{aligned}$$

where the sequence $\{\mathcal{T}_{U,a}^{-1}(C_{j,k})\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ is a frame for $L^2(0,1)$. The sequence $\{\overline{g_j(w)} e^{2\pi i rmw}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ is a Bessel sequence in $L^2(0,1)$ since $\beta_{\mathbb{G}} < \infty$, and satisfying the above expansion in $L^2(0,1)$. According to [8, Lemma 5.6.2], the sequences $\{\mathcal{T}_{U,a}^{-1}(C_{j,k})\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ and $\{\overline{g_j(w)} e^{2\pi i rkw}\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ form a pair of dual frames in $L^2(0,1)$; in particular, by using Lemma 2.2 we obtain that $\alpha_{\mathbb{G}} > 0$ which concludes the proof. □

In case the functions g_j, $j = 1, 2, \dots, s$ are continuous on $\mathbb{R}$, condition (a) in Theorem 3.1 can be expressed in terms of the rank of the matrix $\mathbb{G}(w)$; notice that this occurs, for example, whenever the sequences $\{\mathcal{L}_j a(k)\}_{k\in\mathbb{Z}}$, $j = 1, 2, \dots, s$, belong to $\ell^1(\mathbb{Z})$.

Corollary 3.2. *Assume that the* 1*-periodic extension of the functions* g_j, $j = 1, 2, \dots, s$, *given in* (2.4) *are continuous on* $\mathbb{R}$. *Then, the following conditions are* *equivalent*:

(i) $\operatorname{rank}\mathbb{G}(w) = r$ *for all* $w \in \mathbb{R}$.
(ii) *There exist* $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, *such that the sequence* $\{U^{rk}c_j\}_{k\in\mathbb{Z};j=1,2,\ldots,s}$ *is a frame for* $\mathcal{A}_a$, *and the sampling formula* (3.5) *holds for each* $x \in \mathcal{A}_a$.

Proof. Whenever the functions g_j, $j=1,2,\ldots,s$, are continuous on $\mathbb{R}$, the condition $\alpha_{\mathbb{G}} > 0$ is equivalent to $\det[\mathbb{G}^*(w)\mathbb{G}(w)] \neq 0$ for all $w \in \mathbb{R}$. Indeed, if $\det \mathbb{G}^*(w) \times \mathbb{G}(w) > 0$ then the first row of the matrix $\mathbb{G}^\dagger(w) := [\mathbb{G}^*(w)\mathbb{G}(w)]^{-1}\mathbb{G}^*(w)$, gives a vector $[h_1, h_2, \ldots, h_s]$ satisfying the statement (b) in Theorem 3.1 and, as a consequence, $\alpha_{\mathbb{G}} > 0$. The converse follows from the fact that $\det[\mathbb{G}^*(w)\mathbb{G}(w)] \geq \alpha_{\mathbb{G}}^r$ for all $w \in \mathbb{R}$. Since, $\det[\mathbb{G}^*(w)\mathbb{G}(w)] \neq 0$ is equivalent to $\operatorname{rank}\mathbb{G}(w) = r$ for all $w \in \mathbb{R}$, the result is a consequence of Theorem 3.1. □

Whenever the sampling period r equals the number of U-systems s we are in the presence of Riesz bases, and there exists a unique sampling expansion in Theorem 3.1.

Corollary 3.3. *Let* $b_j \in \mathcal{H}$ *for* $j = 1, 2, \ldots, r$, *i.e.* $r = s$ *in Theorem* 3.1. *Let* $\mathcal{L}_j$ *be its associated* U*-system for* $j = 1, 2, \ldots, r$. *Assume that the function* g_j, $j = 1, 2, \ldots, r$, *given in* (2.4) *belongs to* $L^\infty(0, 1)$; *or equivalently,* $\beta_{\mathbb{G}} < \infty$ *for the associated* $r \times r$ *matrix* $\mathbb{G}(w)$. *The following statements are equivalent*:

(a) $\alpha_{\mathbb{G}} > 0$.
(b) *There exists a Riesz basis* $\{C_{j,k}\}_{k\in\mathbb{Z};j=1,2,\ldots,r}$ *such that for any* $x \in \mathcal{A}_a$ *the expansion*

$$x = \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk) C_{j,k} \quad \text{in } \mathcal{H} \tag{3.6}$$

holds.

In case the equivalent conditions are satisfied, necessarily there exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, r$, *such that* $C_{j,k} = U^{rk}c_j$ *for* $k \in \mathbb{Z}$ *and* $j = 1, 2, \ldots, r$. *Moreover, the interpolation property* $\mathcal{L}_{j'}c_j(rk) = \delta_{j,j'}\delta_{k,0}$, *where* $k \in \mathbb{Z}$ *and* $j, j' = 1, 2, \ldots, r$, *holds.*

Proof. Assume that $\alpha_{\mathbb{G}} > 0$; since $\mathbb{G}(w)$ is a square matrix, this implies that

$$\operatorname*{ess\,inf}_{w\in\mathbb{R}} |\det \mathbb{G}(w)| > 0.$$

Thus, the unique solution $[h_1(w), h_2(w), \ldots, h_r(w)]$ of (3.1) with $h_j \in L^\infty(0, 1)$ for $j = 1, 2, \ldots, r$ is given by the first row of the matrix $\mathbb{G}^{-1}(w)$. According to Theorem 3.1, the sequence $\{C_{j,k}\}_{k\in\mathbb{Z};j=1,2,\ldots,r} := \{U^{rk}c_j\}_{k\in\mathbb{Z};j=1,2,\ldots,r}$, where $c_j = \mathcal{T}_{U,a}(rh_j)$, satisfies the sampling formula (3.6). Moreover, the sequence $\{rh_j(w)\mathrm{e}^{2\pi i rkw}\}_{k\in\mathbb{Z};j=1,2,\ldots,r} = \{\mathcal{T}_{U,a}^{-1}(U^{rk}c_j)\}_{k\in\mathbb{Z};j=1,2,\ldots,r}$ is a frame for $L^2(0, 1)$.

Since $r = s$, according to Lemma 2.2, it is a Riesz basis. Hence, $\{U^{rk}c_j\}_{k\in\mathbb{Z};j=1,2,\dots,r}$ is a Riesz basis for $\mathcal{A}_a$ and (b) is proved.

Conversely, assume now that $\{C_{j,k}\}_{k\in\mathbb{Z};j=1,2,\dots,r}$ is a Riesz basis for $\mathcal{A}_a$ satisfying (3.6). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition $(\mathcal{L}_{j'}C_{j,k})(rk') = \delta_{j,j'}\delta_{k,k'}$ holds for $j, j' = 1, 2, \dots, r$ and $k, k' \in \mathbb{Z}$. Since $\mathcal{T}_{U,a}^{-1}$ is an isomorphism, the sequence $\{\mathcal{T}_{U,a}^{-1}(C_{j,k})\}_{k\in\mathbb{Z};j=1,2,\dots,r}$ is a Riesz basis for $L^2(0,1)$. Expanding the function $\overline{g_{j'}(w)}\mathrm{e}^{-2\pi \mathrm{i} rk'w}$ with respect to the dual basis of $\{\mathcal{T}_{U,a}^{-1}(C_{j,k})\}_{k\in\mathbb{Z};j=1,2,\dots,r}$, denoted by $\{D_{j,k}\}_{k\in\mathbb{Z};j=1,2,\dots,r}$, and having in mind (2.3) we obtain

$$\begin{aligned}\overline{g_{j'}(w)}\mathrm{e}^{2\pi \mathrm{i} rk'w} &= \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} \langle \overline{g_{j'}(\cdot)}\mathrm{e}^{2\pi \mathrm{i} rk'\cdot}, \mathcal{T}_{U,a}^{-1}(C_{j,k})\rangle_{L^2(0,1)} D_{j,k}(w) \\ &= \sum_{k\in\mathbb{Z}} \overline{\mathcal{L}_{j'}C_{j,k}(rk')} D_{j,k}(w) = D_{j',k'}(w).\end{aligned}$$

Therefore, the sequence $\{\overline{g_j(w)}\mathrm{e}^{2\pi \mathrm{i} rkw}\}_{k\in\mathbb{Z};j=1,2,\dots,r}$ is the dual basis of the Riesz basis $\{\mathcal{T}_{U,a}^{-1}(C_{j,k})\}_{k\in\mathbb{Z};j=1,2,\dots,r}$. In particular, it is a Riesz basis for $L^2(0,1)$, which implies, according to Lemma 2.2, that $\alpha_{\mathbb{G}} > 0$, i.e. condition (a). Moreover, the sequence $\{\mathcal{T}_{U,a}^{-1}(C_{j,k})\}_{k\in\mathbb{Z};j=1,2,\dots,r}$ is necessarily the unique dual basis of the Riesz basis $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rkw}\}_{k\in\mathbb{Z};j=1,2,\dots,r}$. Therefore, this proves the uniqueness of the Riesz basis $\{C_{j,k}\}_{k\in\mathbb{Z};j=1,2,\dots,r}$ for $\mathcal{A}_a$ satisfying (3.6). □

Some comments on the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$

Concerning Theorem 3.1, more can be said about the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$, where the vectors $b_j \in \mathcal{H}$ define the U-systems $\mathcal{L}_j \equiv \mathcal{L}_{b_j}$, $j = 1, 2, \dots, s$. Having in mind (2.3) and the isomorphism $\mathcal{T}_{U,a}$, we obtain that

$$\begin{aligned}\frac{\alpha_{\mathbb{G}}}{r}\|\mathcal{T}_{U,a}\|^{-2}\|x\|^2 &\le \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} |\langle x, U^{rk}b_j\rangle|^2 \\ &\le \frac{\beta_{\mathbb{G}}}{r}\|\mathcal{T}_{U,a}^{-1}\|^2\|x\|^2 \quad \text{for all } x \in \mathcal{A}_a. \end{aligned} \tag{3.7}$$

- In case that $b_j \in \mathcal{A}_a$ for each $j = 1, 2, \dots, s$, we derive that $\{U^{rk}b_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$, and it is dual to the frame $\{U^{rk}c_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ in $\mathcal{A}_a$. Thus, the sampling expansion (3.5) is nothing but a frame expansion in $\mathcal{A}_a$.
- In case that some $b_j \notin \mathcal{A}_a$, the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ is not contained in $\mathcal{A}_a$. However, inequalities (3.7) hold. Therefore, the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ is a pseudo-dual frame for the frame $\{U^{rk}c_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ in $\mathcal{A}_a$ (see [20, 21]). Denoting by $P_{\mathcal{A}_a}$ the orthogonal projection onto $\mathcal{A}_a$, we derive from (3.7) that the sequence $\{P_{\mathcal{A}_a}(U^{rk}b_j)\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ is a dual frame of $\{U^{rk}c_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ in $\mathcal{A}_a$.
- Whenever $r = s$, according to the above cases, the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};j=1,2,\dots,s}$ is a Riesz basis or a pseudo-Riesz basis for $\mathcal{A}_a$.

Sampling formulas with prescribed properties

The sampling formula (3.5) can be thought as a filter-bank. Indeed, assume that for $j = 1, 2, \ldots, s$ we have

$$c_{j,h} = \mathcal{T}_{U,a}(rh_j) = r \sum_{n\in\mathbb{Z}} \widehat{h}_j(n) U^n a \quad \text{where } \widehat{h}_j(n) = \int_0^1 h_j(w) \mathrm{e}^{-2\pi i n w} dw, \ \ n \in \mathbb{Z}.$$

Substituting in (3.5), after the change of summation index $m := rk + n$ we obtain

$$x = \sum_{m\in\mathbb{Z}} \left\{ \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} r\mathcal{L}_j x(rk) \widehat{h}_j(m - rk) \right\} U^m a,$$

that is, the relevant data is the output of a filter-bank:

$$\alpha_m := \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} r\mathcal{L}_j x(rk) \widehat{h}_j(m - rk), \quad m \in \mathbb{Z}$$

where the input is the given samples and the impulse responses depend on the sampling vectors $c_{j,h}$, $j = 1, 2, \ldots, s$. In the oversampling setting, i.e. $s > r$, according to (3.3) there exist infinitely many sampling vectors $c_{j,h}$, $j = 1, 2, \ldots, s$, for which the sampling formula (3.5) holds. A natural question is whether we can choose the sampling vectors $c_{j,h}$, $j = 1, 2, \ldots, s$, with prescribed properties.

For instance, a challenging problem is to ask under what conditions we are in the presence of a finite impulse response filter-bank; i.e. $c_{j,h} = r \sum_{\text{finite}} \widehat{h}_j(n) U^n a$, $j = 1, 2, \ldots, s$, or equivalently, when the functions h_j, $j = 1, \ldots, s$, are 2π-periodic trigonometric polynomials. Instead, we deal with Laurent polynomials by using the variable $z = \mathrm{e}^{2\pi i w}$, that is, $\mathsf{g}_j(z) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k) z^k$, $j = 1, 2, \ldots, s$. We introduce the $s \times r$ matrix

$$\mathsf{G}(z) := \begin{bmatrix} \mathsf{g}_1(z) & \mathsf{g}_1(zW) & \cdots & \mathsf{g}_1(zW^{r-1}) \\ \mathsf{g}_2(z) & \mathsf{g}_2(zW) & \cdots & \mathsf{g}_2(zW^{r-1}) \\ \vdots & \vdots & & \vdots \\ \mathsf{g}_s(z) & \mathsf{g}_s(zW) & \cdots & \mathsf{g}_s(zW^{r-1}) \end{bmatrix} = \left[\mathsf{g}_j(zW^k)\right]_{\substack{j=1,2,\ldots,s \\ k=0,1,\ldots,r-1}},$$

where $W : \mathrm{e}^{2\pi i/r}$. In case the functions $\mathsf{g}_j(z)$, $j = 1, 2, \ldots, s$, are Laurent polynomials, the matrix $\mathsf{G}(z)$ has Laurent polynomials entries. Besides, the relationship $\mathbb{G}(w) = \mathsf{G}(\mathrm{e}^{2\pi i w})$, $w \in (0, 1)$, holds.

So that, we are interested in finding Laurent polynomials $\mathsf{h}_j(z)$, $j = 1, 2 \ldots, s$, satisfying

$$[\mathsf{h}_1(z), \mathsf{h}_2(z), \ldots, \mathsf{h}_s(z)] \mathsf{G}(z) = [1, 0, \ldots, 0].$$

Thus, the trigonometric polynomials $h_j(w) := \mathsf{h}_j(\mathrm{e}^{2\pi i w})$, $j = 1, 2, \ldots, s$, satisfy (3.1), and the corresponding reconstruction vectors $c_{j,h} = \mathcal{T}_{U,a}(rh_j)$, $j = 1, 2, \ldots, s$, can be expanded in $\mathcal{A}_a$ with just a finite number of terms. Namely,

$$c_{j,h} = r \sum_{\text{finite}} \widehat{h}_j(n) U^n a, \quad \text{where } \mathsf{h}_j(z) = \sum_{\text{finite}} \widehat{h}_j(n) z^n, \quad j = 1, 2, \ldots, s.$$

The following result holds:

Theorem 3.4. *Assume that the sequences* $\{\mathcal{L}_j a(k)\}_{k\in\mathbb{Z}}$, $j = 1, 2, \ldots, s$, *contain only a finite number of nonzero terms. Then, there exists a vector* $\mathsf{h}(z) := [\mathsf{h}_1(z), \mathsf{h}_2(z), \ldots, \mathsf{h}_s(z)]$ *whose entries are Laurent polynomials, and satisfying* $\mathsf{h}(z)\mathsf{G}(z) = [1, 0, \ldots, 0]$ *if and only if*

$$\operatorname{rank} \mathsf{G}(z) = r \quad \textit{for all} \ \ z \in \mathbb{C}\backslash\{0\}.$$

Proof. This result is a consequence of the next lemma which proof can be found in [34, Theorems 5.1 and 5.6]. □

Lemma 3.5. *Let* $\mathsf{G}(z)$ *be an* $s \times r$ *matrix whose entries are Laurent polynomials. Then, there exists an* $r \times s$ *matrix* $\mathsf{H}(z)$ *whose entries are also Laurent polynomials satisfying* $\mathsf{H}(z)\mathsf{G}(z) = \mathbb{I}_r$ *if and only if* $\operatorname{rank} \mathsf{G}(z) = r$ *for all* $z \in \mathbb{C}\backslash\{0\}$.

Analogously we can consider the case where the coefficients of the reconstruction vectors $c_{j,h} = r\sum_{n\in\mathbb{Z}} \widehat{h}_j(n) U^n a$, $j = 1, 2, \ldots, s$, have exponential decay, i.e. there exist $C > 0$ and $q \in (0, 1)$ such that $|\widehat{h}_j(n)| \le C q^{|n|}$, $n \in \mathbb{Z}$, $j = 1, 2, \ldots, s$. Assuming that the sequences $\{\mathcal{L}_j a(k)\}_{k\in\mathbb{Z}}$, $j = 1, 2, \ldots, s$, have exponential decay then, we can find reconstruction vectors $c_{j,h}$ such that the sequences $\{\widehat{h}_j(n)\}_{n\in\mathbb{Z}}$, $j = 1, 2, \ldots, s$, have exponential decay if and only if $\operatorname{rank} \mathsf{G}(z) = r$ for all $z \in \mathbb{C}$ such that $|z| = 1$. For the details, see [16] and references therein.

4. Time-Jitter Error: Irregular Sampling in $\mathcal{A}_a$

A close look to Sec. 3 shows that all the regular sampling results have been proved without the formalism of a continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ in $\mathcal{H}$: we have only used the integer powers $\{U^n\}_{n\in\mathbb{Z}}$ which are completely determined from the unitary operator U. However, if we are concerned with the jitter error in a sampling formula as (3.5), the group of unitary operators becomes essential. Here, we dispose of a perturbed sequence of samples $\{(\mathcal{L}_j x)(rm + \epsilon_{mj})\}_{m\in\mathbb{Z}; j=1,2,\ldots,s}$, with errors $\epsilon_{mj} \in \mathbb{R}$, for the recovery of $x \in \mathcal{A}_a$. By using (2.3) and (2.2) we obtain:

$$\mathcal{L}_j x(rm) = \langle F, \overline{g_j(w)} \mathrm{e}^{2\pi i r m w} \rangle_{L^2(0,1)} \quad \text{and}$$

$$\mathcal{L}_j x(rm + \epsilon_{mj}) = \langle F, \overline{g_{m,j}(w)} \mathrm{e}^{2\pi i r m w} \rangle_{L^2(0,1)},$$

where the functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k) \mathrm{e}^{2\pi i k w} \quad \text{and} \quad g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k + \epsilon_{mj}) \mathrm{e}^{2\pi i k w},$$

belong to $L^2(0, 1)$. Let $\mathbb{G}(w)$ be the $s \times r$ matrix given in (2.5), associated with the functions g_j, $j = 1, 2, \ldots, s$. In the case that $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$, the sequence

$\{\overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ with optimal frame bounds $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$. Thus, as in [15], we can see the sequence $\{\overline{g_{m,j}(w)}\mathrm{e}^{2\pi i r m w}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ in $L^2(0,1)$ as a perturbation of the frame $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ in $L^2(0,1)$. The following result on frame perturbation, which proof can be found in [8, p. 354] will be used later.

Lemma 4.1. *Let $\{x_n\}_{n=1}^\infty$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A, B, and let $\{y_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R<A$ such that*

$$\sum_{n=1}^\infty |\langle x_n - y_n, x\rangle|^2 \le R\|x\|^2 \quad \textit{for each } x\in\mathcal{H},$$

then the sequence $\{y_n\}_{n=1}^\infty$ is also a frame for $\mathcal{H}$ with bounds $A(1-\sqrt{R/A})^2$ and $B(1+\sqrt{R/B})^2$. If the sequence $\{x_n\}_{n=1}^\infty$ is a Riesz basis, then the sequence $\{y_n\}_{n=1}^\infty$ is also a Riesz basis.

The time-jitter error sampling expansion

Given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$, assume that the operator

$$\begin{aligned} D_{\boldsymbol{\epsilon}} &: \ell^2(\mathbb{Z}) \longrightarrow \ell^2_s(\mathbb{Z}),\\ c = \{c_l\}_{l\in\mathbb{Z}} &\mapsto D_{\boldsymbol{\epsilon}} c := (D_{\boldsymbol{\epsilon},1}c, \dots, D_{\boldsymbol{\epsilon},s}c), \end{aligned}$$

is well-defined, where, for $j=1,2,\dots,s$,

$$D_{\boldsymbol{\epsilon},j}c := \left\{\sum_{k\in\mathbb{Z}} [\mathcal{L}_j a(rm-k+\epsilon_{mj}) - \mathcal{L}_j a(rm-k)]c_k\right\}_{m\in\mathbb{Z}}. \tag{4.1}$$

The operator norm (it could be infinity) is defined as usual

$$\|D_{\boldsymbol{\epsilon}}\| := \sup_{c\in\ell^2(\mathbb{Z})\setminus\{0\}} \frac{\|D_{\boldsymbol{\epsilon}}c\|_{\ell^2_s(\mathbb{Z})}}{\|c\|_{\ell^2(\mathbb{Z})}},$$

where $\|D_\epsilon c\|^2_{\ell^2_s(\mathbb{Z})} := \sum_{j=1}^s \|D_{\epsilon,j}c\|^2_{\ell^2(\mathbb{Z})}$ for each $c\in\ell^2(\mathbb{Z})$.

Theorem 4.2. *Assume that for the functions g_j, $j=1,2,\dots,s$, given in (2.4) we have $0<\alpha_{\mathbb{G}}\le\beta_{\mathbb{G}}<\infty$. Let $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ be an error sequence satisfying the inequality $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$. Then, there exists a frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for any $x\in\mathcal{A}_a$, the sampling expansion*

$$x = \sum_{j=1}^s \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm+\epsilon_{mj}) C^{\boldsymbol{\epsilon}}_{j,m} \quad \textit{in } \mathcal{H}, \tag{4.2}$$

holds. Moreover, when $r=s$ the sequence $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ is a Riesz basis for $\mathcal{A}_a$, and the interpolation property $(\mathcal{L}_l C^{\boldsymbol{\epsilon}}_{j,n})(rm+\epsilon_{mj}) = \delta_{j,l}\delta_{n,m}$ holds.

Proof. The sequence $\{\overline{g_j(w)}\mathrm{e}^{2\pi\mathrm{i}rmw}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ is a frame (a Riesz basis if $r=s$) for $L^2(0,1)$ with optimal frame (Riesz) bounds $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$. For any $F(w)=\sum_{l\in\mathbb{Z}} a_l \mathrm{e}^{2\pi\mathrm{i}lw}$ in $L^2(0,1)$ we have

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s} |\langle \overline{g_{m,j}(\cdot)}\mathrm{e}^{2\pi\mathrm{i}rm\cdot} - \overline{g_j(\cdot)}\mathrm{e}^{2\pi\mathrm{i}rm\cdot}, F(\cdot)\rangle_{L^2(0,1)}|^2 \\
&\quad = \sum_{m\in\mathbb{Z}}\sum_{j=1}^{s} \left| \left\langle \sum_{k\in\mathbb{Z}} (\overline{\mathcal{L}_j a(k+\epsilon_{mj})} - \overline{\mathcal{L}_j a(k)})\mathrm{e}^{2\pi\mathrm{i}(rm-k)\cdot}, F(\cdot) \right\rangle_{L^2(0,1)} \right|^2 \\
&\quad = \sum_{m\in\mathbb{Z}}\sum_{j=1}^{s} \left| \left\langle \sum_{k\in\mathbb{Z}} (\overline{\mathcal{L}_j a(rm-k+\epsilon_{mj})} - \overline{\mathcal{L}_j a(rm-k)})\mathrm{e}^{2\pi\mathrm{i}k\cdot}, F(\cdot) \right\rangle_{L^2(0,1)} \right|^2 \\
&\quad = \sum_{m\in\mathbb{Z}}\sum_{j=1}^{s} \left| \sum_{k\in\mathbb{Z}} (\overline{\mathcal{L}_j a(rm-k+\epsilon_{mj})} - \overline{\mathcal{L}_j a(rm-k)})\overline{a}_k \right|^2 \\
&\quad = \sum_{j=1}^{s} \|D_{\epsilon,j}\{a_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})} \\
&\quad \le \|D_{\boldsymbol{\epsilon}}\|^2 \|\{a_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})} \\
&\quad = \|D_{\boldsymbol{\epsilon}}\|^2 \|F\|^2_{L^2(0,1)}.
\end{aligned} \tag{4.3}
$$

By using Lemma 4.1 we obtain that the sequence $\{\overline{g_{m,j}(w)}\mathrm{e}^{2\pi\mathrm{i}rmw}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ (a Riesz basis if $r=s$). Let $\{h^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ be its canonical dual frame. Hence, for any $F\in L^2(0,1)$

$$
\begin{aligned}
F &= \sum_{m\in\mathbb{Z}}\sum_{j=1}^{s} \langle F(\cdot), \overline{g_{m,j}(\cdot)}\mathrm{e}^{2\pi\mathrm{i}rm\cdot}\rangle_{L^2(0,1)} h^{\boldsymbol{\epsilon}}_{j,m} \\
&= \sum_{m\in\mathbb{Z}}\sum_{j=1}^{s} \mathcal{L}_j x(rm+\epsilon_{mj}) h^{\boldsymbol{\epsilon}}_{j,m} \quad \text{in } L^2(0,1).
\end{aligned}
$$

Applying the isomorphism $\mathcal{T}_{U,a}$, one gets (4.2), where $C^{\boldsymbol{\epsilon}}_{j,m} := \mathcal{T}_{U,a}(h^{\boldsymbol{\epsilon}}_{j,m})$ for $m\in\mathbb{Z}$ and $j=1,2,\dots,s$. Since $\mathcal{T}_{U,a}$ is an isomorphism between $L^2(0,1)$ and $\mathcal{A}_a$, the sequence $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$ (a Riesz basis if $r=s$). The interpolatory property in the case $r=s$ follows from the uniqueness of the coefficients with respect to a Riesz basis. □

Sampling formula (4.2) is useless from a practical point of view: it is impossible to determine the involved frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$. As a consequence, in order to recover $x\in\mathcal{A}_a$ from the sequence of samples $\{(\mathcal{L}_j x)(rm+\epsilon_{mj})\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ we should implement a frame algorithm in $\ell^2(\mathbb{Z})$ (see [15]); another possibility is given in [1].

In order to prove the existence of sequences $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};j=1,\dots,s}$ such that $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$ we need some results from the group of unitary operators theory.

A brief excursion on groups of unitary operators

Let $\{U^t\}_{t\in\mathbb{R}}$ denote a continuous *group of unitary operators* in $\mathcal{H}$. Classical Stone's theorem [26] assures us the existence of a self-adjoint operator T (may be unbounded) such that $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$. This self-adjoint operator T, defined on the dense domain of $\mathcal{H}$

$$D_T := \left\{x \in \mathcal{H} \text{ such that} \int_{-\infty}^{\infty} w^2 d\|E_w x\|^2 < \infty\right\},$$

admits the *spectral representation* $T = \int_{-\infty}^{\infty} w dE_w$ which means:

$$\langle Tx, y\rangle = \int_{-\infty}^{\infty} wd\langle E_w x, y\rangle \quad \text{for any } x \in D_T \text{ and } y \in \mathcal{H},$$

where $\{E_w\}_{w\in\mathbb{R}}$ is the corresponding *resolution of the identity*, i.e. a one-parameter family of projection operators E_w in $\mathcal{H}$ such that

(i) $E_{-\infty} := \lim_{w\to-\infty} E_w = O_{\mathcal{H}}$, $E_\infty := \lim_{w\to\infty} E_w = I_{\mathcal{H}}$,
(ii) $E_{w^-} = E_w$ for any $-\infty < w < \infty$,
(iii) $E_u E_v = E_w$ where $w = \min\{u, v\}$.

Recall that $\|E_w x\|^2$ and $\langle E_w x, y\rangle$, as functions of w, have bounded variation and define, respectively, a positive and a complex Borel measure on $\mathbb{R}$.

Furthermore, for any $x \in D_T$ we have that $\lim_{t\to 0} \frac{U^t x - x}{t} = \mathrm{i}\,Tx$ and the operator $\mathrm{i}T$ is said to be the *infinitesimal generator* of the group $\{U^t\}_{t\in\mathbb{R}}$. For each $x \in D_T$, $U^t x$ is a continuous differentiable function of t. Notice that, whenever the self-adjoint operator T is bounded, $D_T = \mathcal{H}$ and $\mathrm{e}^{\mathrm{i}tT}$ can be defined as the usual exponential series; in any case, $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$ means that

$$\langle U^t x, y\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}wt} d\langle E_w x, y\rangle, \quad t \in \mathbb{R},$$

where $x \in D_T$ and $y \in \mathcal{H}$.

Finally, a comment on the continuity of a group of unitary operators: The group is said to be *strongly continuous* if, for each $x \in \mathcal{H}$ and $t_0 \in \mathbb{R}$, $U^t x \to U^{t_0} x$ as $t \to t_0$. If $\mathcal{H}$ is a separable Hilbert space, strong continuity can be deduced from continuity and even from weak measurability, i.e. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a Lebesgue measurable function of t for any $x, y \in \mathcal{H}$. See, for instance, [2, 7, 32, 33].

On the existence of sequences $\boldsymbol{\epsilon}$ such that $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$

Assuming that $b_j \in D_T$, $j = 1, 2, \dots, s$, the functions $\mathcal{L}_j a(t)$, $j = 1, 2, \dots, s$, are continuously differentiable on $\mathbb{R}$. If, for instance, we demand in addition that, for each $j = 1, 2, \ldots, s$, there exists $\eta_j > 0$ such that

$$(\mathcal{L}_j a)'(t) = O(|t|^{-(1+\eta_j)}) \quad \text{whenever } |t| \to \infty, \tag{4.4}$$

then we can find out a finite bound for the norm $\|D_{\boldsymbol{\epsilon}}\|^2$. Indeed, for $j = 1, 2, \ldots, s$ and $n, m \in \mathbb{Z}$ denote

$$d_{m,k}^{(j)} := \mathcal{L}_j a(rm - k + \epsilon_{m,j}) - \mathcal{L}_j a(rm - k).$$

Taking into account (4.1), for any sequence $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned}
\|D_\epsilon c\|^2_{\ell^2_s(\mathbb{Z})} &= \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} d_{m,k}^{(j)} c_k \right|^2 \\
&\le \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \sum_{l,k\in\mathbb{Z}} |d_{m,l}^{(j)} c_l \overline{d}_{m,k}^{(j)} \overline{c}_k| \\
&= \sum_{j=1}^{s} \sum_{l,k\in\mathbb{Z}} |c_l|\,|c_k| \sum_{m\in\mathbb{Z}} |d_{m,l}^{(j)} d_{m,k}^{(j)}| \\
&\le \sum_{j=1}^{s} \sum_{l,k\in\mathbb{Z}} \frac{|c_l|^2 + |c_k|^2}{2} \sum_{m\in\mathbb{Z}} |d_{m,l}^{(j)} d_{m,k}^{(j)}| \\
&= \sum_{j=1}^{s} \sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{k,m\in\mathbb{Z}} |d_{m,l}^{(j)} d_{m,k}^{(j)}|.
\end{aligned} \tag{4.5}$$

Under the decay conditions (4.4), for $|\gamma| \le 1/2$ we define the continuous functions,

$$M_{(\mathcal{L}_j a)'}(\gamma) := \sum_{k\in\mathbb{Z}} \max_{t\in[k-\gamma, k+\gamma]} |(\mathcal{L}_j a)'(t)|,$$

and

$$N_{(\mathcal{L}_j a)'}(\gamma) := \max_{k=0,1,\ldots,r-1} \sum_{m\in\mathbb{Z}} \max_{t\in[rm+k-\gamma, rm+k+\gamma]} |(\mathcal{L}_j a)'(t)|.$$

Notice that $N_{(\mathcal{L}_j a)'}(\gamma) \le M_{(\mathcal{L}_j a)'}(\gamma)$ and for $r = 1$ the equality holds.

Theorem 4.3. *Given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z}; j=1,\ldots,s}$, define the constant $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}|$ for each $j = 1, 2, \ldots, s$. Then, the inequality*

$$\|D_{\boldsymbol{\epsilon}}\|^2 \le \sum_{j=1}^{s} \gamma_j^2 N_{(\mathcal{L}_j a)'}(\gamma_j) M_{(\mathcal{L}_j a)'}(\gamma_j)$$

holds and, as a consequence, condition

$$\sum_{j=1}^{s} \gamma_j^2 N_{(\mathcal{L}_j a)'}(\gamma_j) M_{(\mathcal{L}_j a)'}(\gamma_j) < \frac{\alpha_{\mathbf{G}}}{r}$$

ensures the hypothesis $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$ in Theorem 4.2.

Proof. For each $j = 1, 2, \ldots, s$, the mean value theorem gives

$$\sup_{d\in[-\gamma_j,\gamma_j]} \sum_{n\in\mathbb{Z}} |\mathcal{L}_j a(n+d) - \mathcal{L}_j a(n)| \leq \gamma_j M_{(\mathcal{L}_j a)'}(\gamma_j), \tag{4.6}$$

and

$$\sup_{\substack{k=0,1,\ldots,r-1\\ \{d_n\}\subset[-\gamma_j,\gamma_j]}} \sum_{n\in\mathbb{Z}} |\mathcal{L}_j a(rn+k+d_n) - \mathcal{L}_j a(rn+k)| \leq \gamma_j N_{(\mathcal{L}_j a)'}(\gamma_j). \tag{4.7}$$

Thus, using (4.6) and (4.7), inequality (4.5) becomes

$$\begin{aligned}
\|D_\epsilon c\|^2_{\ell^2_s(\mathbb{Z})} &\leq \sum_{j=1}^{s}\sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{k,m\in\mathbb{Z}} |d^{(j)}_{m,l} d^{(j)}_{m,k}| \\
&\leq \sum_{j=1}^{s}\sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{m\in\mathbb{Z}} |d^{(j)}_{m,l}| \gamma_j M_{(\mathcal{L}_j a)'}(\gamma_j) \\
&\leq \sum_{j=1}^{s}\sum_{l\in\mathbb{Z}} |c_l|^2 (\gamma_j)^2 M_{(\mathcal{L}_j a)'}(\gamma_j) N_{(\mathcal{L}_j a)'}(\gamma_j) \\
&= \|c\|^2_{\ell^2(\mathbb{Z})} \sum_{j=1}^{s} \gamma_j^2 N_{(\mathcal{L}_j a)'}(\gamma_j) M_{(\mathcal{L}_j a)'}(\gamma_j),
\end{aligned}$$

which concludes the proof. □

5. The Case of Multiple Generators

The case of L generators can be analogously derived. Indeed, consider the U-invariant subspace generated by $\mathbf{a} := \{a_1, a_2, \ldots, a_L\} \subset \mathcal{H}$, i.e.

$$\mathcal{A}_{\mathbf{a}} := \overline{\operatorname{span}}\{U^n a_l, n \in \mathbb{Z}; l = 1, 2\ldots, L\}.$$

Assuming that the sequence $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ is a Riesz sequence in $\mathcal{H}$, the U-invariant subspace $\mathcal{A}_{\mathbf{a}}$ can be expressed as

$$\mathcal{A}_{\mathbf{a}} = \left\{ \sum_{l=1}^{L}\sum_{n\in\mathbb{Z}} \alpha^l_n U^n a : \{\alpha^l_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}); l = 1, 2\ldots, L \right\}.$$

The sequence $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ can be thought as an L-dimensional stationary sequence. Its *covariance matrix* $\mathbf{R}_{\mathbf{a}}(k)$ is the $L \times L$ matrix

$$\mathbf{R}_{\mathbf{a}}(k) := [\langle U^k a_m, a_n\rangle_{\mathcal{H}}]_{m,n=1,2,\ldots,L}, \quad k \in \mathbb{Z}.$$

It admits the spectral representation [19]:

$$\mathbf{R}_{\mathbf{a}}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\boldsymbol{\mu}_{\mathbf{a}}(\theta), \quad k \in \mathbb{Z}.$$

The *spectral measure* $\boldsymbol{\mu}_{\mathbf{a}}$ is an $L \times L$ matrix; its entries are the spectral measures associated with the cross-correlation functions $R_{m,n}(k) := \langle U^k a_m, a_n\rangle_{\mathcal{H}}$. It can be decomposed into an absolute continuous part and its singular part. Thus we can write

$$d\boldsymbol{\mu}_{\mathbf{a}}(\theta) = \boldsymbol{\Phi}_{\mathbf{a}}(\theta)d\theta + d\boldsymbol{\mu}^s_{\mathbf{a}}(\theta).$$

In case that the singular part $\boldsymbol{\mu}^s_{\mathbf{a}} \equiv 0$, the hermitian $L \times L$ matrix $\boldsymbol{\Phi}_{\mathbf{a}}(\theta)$ is called the *spectral density* of the sequence $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$. The following theorem holds.

Theorem 5.1. *Let $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ be a sequence obtained from a unitary operator in a separable Hilbert space $\mathcal{H}$ with spectral measure $d\boldsymbol{\mu}_{\mathbf{a}}(\theta) = \boldsymbol{\Phi}_{\mathbf{a}}(\theta)d\theta + d\boldsymbol{\mu}^s_{\mathbf{a}}(\theta)$, and let $\mathcal{A}_{\mathbf{a}}$ be the closed subspace spanned by $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$. Then the sequence $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$ if and only if the singular part $\boldsymbol{\mu}^s_{\mathbf{a}} \equiv 0$ and*

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \lambda_{\min}[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)] \leq \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \lambda_{\max}[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)] < \infty. \tag{5.1}$$

Proof. For a fixed ℓ^2_L-sequence $c := \{c^l_n\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ we have

$$\begin{aligned}
\left\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c^l_k U^k a_l \right\|^2 &= \sum_{i,j=1}^{L} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^i_m \bar{c}^j_m \langle U^m a_i, U^n a_j \rangle \\
&= \sum_{i,j=1}^{L} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^i_m \bar{c}^j_n \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{im\theta} \mathrm{e}^{-in\theta} d\mu_{a_i,a_j}(\theta) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} (\mathbf{c}_m \,\mathrm{e}^{im\theta})^\top d\boldsymbol{\mu}_{\mathbf{a}}(\theta) \bar{\mathbf{c}}_n \,\mathrm{e}^{-in\theta},
\end{aligned} \tag{5.2}$$

where $\mathbf{c}_k = (c^1_k, c^2_k, \ldots, c^L_k)^\top$ for every $k \in \mathbb{Z}$.

First, we show that if the measure $\boldsymbol{\mu}_{\mathbf{a}}$ is not absolutely continuous with respect to Lebesgue measure λ then $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ is not a Riesz basis for $\mathcal{A}_{\mathbf{a}}$. Indeed, if the spectral measure $\boldsymbol{\mu}_{\mathbf{a}}$ is not absolutely continuous with respect to Lebesgue measure then there exists $i \in \{1, 2, \ldots, L\}$ such that the positive spectral measure μ_{a_i,a_i} is not absolutely continuous with respect to Lebesgue measure; this comes from the fact that, if any spectral measure in the diagonal μ_{a_j,a_j} is absolutely continuous with respect to Lebesgue measure, the same occurs for each measure μ_{a_j,a_k} with $k \neq j$ (see [7, p. 137]). Then, $\mu_{a_i,a_i}(B) > 0$ for a (Lebesgue) measurable set $B \subset (-\pi, \pi)$ of Lebesgue measure zero. Bearing in mind that every measurable set is included in a Borel set, actually an intersection of a countable collection of open sets, having the same Lebesgue measure (see [25, p. 63]), we take B to be a Borel set. Moreover, since every finite Borel measure on $(-\pi, \pi)$ is inner regular (see [25, p. 340]) we may also assume that B is a compact set. For any $\varepsilon > 0$ there exists a sequence of disjoint open intervals $I_j \subset (-\pi, \pi)$ such that

$$B \subset \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda(I_j) \leq \lambda(B) + \varepsilon = \varepsilon,$$

(see [25, pp. 58 and 42]). Since B is compact we may take the sequence to be finite. Hence, for every $N \in \mathbb{N}$ there exist open disjoint intervals $I_1^N, I_2^N, \ldots, I_{j_N}^N$ in $(-\pi, \pi)$ such that

$$B \subset \bigcup_{j=1}^{j_N} I_j^N \quad \text{and} \quad \sum_{j=1}^{j_N} \lambda(I_j^N) \leq \frac{1}{3^N}.$$

Besides, $\sum_{j=1}^{j_N} \mu_{a_i,a_i}(I_j^N) \geq \mu_{a_i,a_i}(B)$. Consider the function $g_N\colon (-\pi,\pi) \to \mathbb{R}$, where $g_N = 2^{N/2}\chi_{\bigcup_{j=1}^{j_N} I_j^N}$, that satisfies

$$\|g_N\|_2^2 = 2^N \sum_{j=1}^{j_N} \lambda(I_j^N) \leq \frac{2^N}{3^N} < 1.$$

We modify and extend each g_N to obtain a 2π-periodic function $f_N\colon \mathbb{R} \to \mathbb{R}$ such that f_N and its derivative are continuous on $\mathbb{R}$, $\|f_N\|_2^2 \leq 1$ and $f_N(\theta) = g_N(\theta)$ for every $\theta \in \bigcup_{j=1}^{j_N} I_j^N$. Let $\sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta}$ be the Fourier series of f_N. First, by using Parseval's identity we have

$$\|c_k^N\|_2^2 = \frac{1}{2\pi}\|f_N\|_2^2 \leq \frac{1}{2\pi} \quad \text{for every } N \in \mathbb{N},$$

so that $\{c^N\}_{N=1}^{\infty}$ is a bounded sequence in $\ell^2(\mathbb{Z})$. Besides, the regularity of each f_N ensures that each Fourier series converges uniformly to f_N. Therefore, each series $\sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta}$ converges to f_N in $L^2_{\mu_{a_i,a_i}(-\pi,\pi)}$ and consequently,

$$\begin{aligned}\Big\|\sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta}\Big\|^2_{L^2_{\mu_{a_i,a_i}}(-\pi,\pi)} &= \int_{-\pi}^{\pi} |f_N|^2 d\mu_{a_i,a_i} \geq \int_{-\pi}^{\pi} |g_N|^2 d\mu_{a_i,a_i} \\ &= 2^N \sum_{j=1}^{j_N} \mu_{a_i,a_i}(I_j^N) \geq 2^N \mu_{a_i,a_i}(B).\end{aligned}$$

For every $c^N \in \ell^2(\mathbb{Z})$ we consider the ℓ^2_L-sequence $\{c_n^{Nl}\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ given by $c_n^{Ni} = c_n^N$ and $c_n^{Nl} = 0$ if $l \neq i$. Substituting each $\{c_n^{Nl}\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ in (5.2) we have that

$$\Big\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} c_k^{Nl} U^k a_l\Big\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k\in\mathbb{Z}} c_k^N \mathrm{e}^{\mathrm{i}k\theta}\Big|^2 d\mu_{a_i,a_i}(\theta)$$

tends to infinity with N, so $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\ldots,L}$ cannot be a Bessel sequence, therefore, not a Riesz basis.

For the remainder of the proof we assume that the singular part $\boldsymbol{\mu}_{\mathbf{a}}^s \equiv 0$ and that $d\boldsymbol{\mu}_{\mathbf{a}}(\theta) = \mathbf{\Phi}_{\mathbf{a}}(\theta)d\theta$. Then (5.2) yields that

$$\Big\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} c_k^l U^k a_l\Big\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\sum_{m\in\mathbb{Z}} \mathbf{c}_m \mathrm{e}^{\mathrm{i}m\theta}\Big)^{\top} \mathbf{\Phi}_{\mathbf{a}}(\theta)\overline{\sum_{n\in\mathbb{Z}} \mathbf{c}_n \mathrm{e}^{\mathrm{i}n\theta}}d\theta. \tag{5.3}$$

We have to show that $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_\mathbf{a}$ if and only if (5.1) holds. Rayleigh–Ritz theorem (see [17, p. 176]) provides the inequalities

$$\lambda_{\min}[\mathbf{\Phi}_\mathbf{a}(\theta)]\left|\sum_{k\in\mathbb{Z}}\mathbf{c}_k\,\mathrm{e}^{\mathrm{i}k\theta}\right|^2 \le \left(\sum_{m\in\mathbb{Z}}\mathbf{c}_m\,\mathrm{e}^{\mathrm{i}m\theta}\right)^\top \mathbf{\Phi}_\mathbf{a}(\theta)\overline{\sum_{n\in\mathbb{Z}}\mathbf{c}_n\mathrm{e}^{\mathrm{i}n\theta}}$$
$$\le \lambda_{\max}[\mathbf{\Phi}_\mathbf{a}(\theta)]\left|\sum_{k\in\mathbb{Z}}\mathbf{c}_k\mathrm{e}^{\mathrm{i}k\theta}\right|^2,$$

and taking into account (5.3) we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda_{\min}[\mathbf{\Phi}_\mathbf{a}(\theta)]\left|\sum_{k\in\mathbb{Z}}\mathbf{c}_k\,\mathrm{e}^{\mathrm{i}k\theta}\right|^2 d\theta \le \left\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}c_k^l U^k a_l\right\|^2$$
$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda_{\max}[\mathbf{\Phi}_\mathbf{a}(\theta)]\left|\sum_{k\in\mathbb{Z}}\mathbf{c}_k\mathrm{e}^{\mathrm{i}k\theta}\right|^2 d\theta,$$

so that

$$\operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)}\lambda_{\min}[\mathbf{\Phi}_\mathbf{a}(\theta)]\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}|c_k^l|^2 \le \left\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}c_k^l U^k a_l\right\|^2$$
$$\le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)}\lambda_{\max}[\mathbf{\Phi}_\mathbf{a}(\theta)]\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}|c_k^l|^2.$$

Therefore, (5.1) implies that $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_\mathbf{a}$.

Conversely, if $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_\mathbf{a}$ then there exist constants $0<A\le B<\infty$ such that

$$A\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}|c_k^l|^2 \le \left\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}c_k^l U^k a_l\right\|^2 \le B\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}|c_k^l|^2 \tag{5.4}$$

for every ℓ^2_L-sequence $c:=\{c_n^l\}_{n\in\mathbb{Z};l=1,2,\dots,L}$. Let us prove that

$$A\le \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)}\lambda_{\min}[\mathbf{\Phi}_\mathbf{a}(\theta)] \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)}\lambda_{\max}[\mathbf{\Phi}_\mathbf{a}(\theta)]\le B. \tag{5.5}$$

Proceeding by contradiction, if (5.5) would not hold, then

$$A\le\lambda_{\min}[\mathbf{\Phi}_\mathbf{a}(\theta)]\le\lambda_{\max}[\mathbf{\Phi}_\mathbf{a}(\theta)]\le B$$

does not hold on a subset of $(-\pi,\pi)$ with positive Lebesgue measure. In case the set $\Gamma_B:=\{\theta\in(-\pi,\pi)\colon \lambda_{\max}[\mathbf{\Phi}_\mathbf{a}(\theta)]>B\}$ has positive Lebesgue measure we introduce the Fourier expansion of the function $F\in L^2_L(-\pi,\pi)$ ($L^2_L(-\pi,\pi)$ denotes the usual product Hilbert space $L^2(-\pi,\pi)\times\cdots\times L^2(-\pi,\pi)(L$ times)) in (5.3), where $F(\theta) = \mathbf{X}(\theta)\chi_{\Gamma_B}(\theta)$ and $\mathbf{X}(\theta)$ is an eigenvector of norm 1 associated with the biggest eigenvalue of $\mathbf{\Phi_a}(\theta)$. We get

$$\left\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \right\|^2 = \frac{1}{2\pi} \int_{\Gamma_B} \lambda_{\max}[\mathbf{\Phi_a}(\theta)] d\theta > \frac{1}{2\pi} \int_{\Gamma_B} B d\theta$$

which contradicts the right inequality in (5.4) for such a Fourier expansion. Whenever Lebesgue measure of the set Γ_B is zero then we proceed in a similar way with the set of positive Lebesgue measure $\Gamma_A := \{\theta \in (-\pi, \pi) \colon \lambda_{\min}[\mathbf{\Phi_a}(\theta)] < A\}$. □

The above proof is similar to that of [24, Lemma 2], except we do not exclude the case in which the singular measure is atomless. Another characterization for being $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\dots,L}$ a Riesz basis for $\mathcal{A}_\mathbf{a}$ can be found in [3].

The resulting regular sampling formulas

As in the one-generator case, the space $\mathcal{A}_\mathbf{a}$ is the image of the usual product Hilbert space $L^2_L(0,1)$ by means of the isomorphism $\mathcal{T}_{U,\mathbf{a}} : L^2_L(0,1) \longrightarrow \mathcal{A}_\mathbf{a}$, which maps the orthonormal basis $\{\mathrm{e}^{-2\pi i n w}\mathbf{e}_l\}_{n\in\mathbb{Z};l=1,2,\dots,L}$ for $L^2_L(0,1)$ (here, $\{\mathbf{e}_l\}_{l=1}^{L}$ denotes the canonical basis for $\mathbb{C}^L$) onto the Riesz basis $\{U^n a_l\}_{n\in\mathbb{Z};l=1,2,\dots,L}$ for $\mathcal{A}_\mathbf{a}$, i.e.

$$\mathcal{T}_{U,\mathbf{a}}\mathbf{F} := \sum_{l=1}^{L} \sum_{n\in\mathbb{Z}} \langle F_l, \mathrm{e}^{2\pi i n \cdot} \rangle_{L^2(0,1)} U^n a_l = \sum_{l=1}^{L} \sum_{n\in\mathbb{Z}} \alpha_n^l U^n a_l, \tag{5.6}$$

where $\mathbf{F} = (F_1, F_2, \dots, F_L)^\top \in L^2_L(0,1)$.

Here, for $\mathbf{F} \in L^2_L(0,1)$ and $N \in \mathbb{Z}$ the U-shift property reads:

$$\mathcal{T}_{U,\mathbf{a}}(\mathbf{F}\mathrm{e}^{2\pi i N w}) = U^N(\mathcal{T}_{U,\mathbf{a}}\mathbf{F}). \tag{5.7}$$

Concerning the representation of an U-system $\mathcal{L}_b$, for $x \in \mathcal{A}_\mathbf{a}$ we have

$$\begin{aligned}\mathcal{L}_b x(t) = \langle x, U^t b\rangle_\mathcal{H} &= \sum_{l=1}^{L} \sum_{n\in\mathbb{Z}} \alpha_n^l \overline{\langle U^t b, U^n a_l\rangle_\mathcal{H}} \\ &= \sum_{l=1}^{L} \left\langle F_l, \sum_{n\in\mathbb{Z}} \langle U^t b, U^n a_l\rangle_\mathcal{H} \mathrm{e}^{2\pi i n w} \right\rangle_{L^2(0,1)} = \langle \mathbf{F}, \mathbf{K}_t\rangle_{L^2_L(0,1)},\end{aligned}$$

where $\mathcal{T}_{U,a}\mathbf{F} = x$, $\mathbf{F} = (F_1, F_2, \dots, F_L)^\top \in L^2_L(0,1)$, and the function

$$\begin{aligned}\mathbf{K}_t(w) := \Bigg(&\sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_b a_1(t-n)} \mathrm{e}^{2\pi i n w}, \sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_b a_2(t-n)} \mathrm{e}^{2\pi i n w}, \dots, \\ &\sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_b a_L(t-n)} \mathrm{e}^{2\pi i n w} \Bigg)^\top\end{aligned}$$

belongs to $L^2_L(0,1)$. In particular, given s U-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$ associated with b_j in $\mathcal{H}$, $j=1,2,\dots,s$, we get the expression for the samples $\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};j=1,2,\dots,s}$:

$$\mathcal{L}_j x(rm) = \langle \mathbf{F}, \overline{\mathbf{g}_j(w)}\mathrm{e}^{2\pi \mathrm{i} rmw}\rangle_{L^2_L(0,1)} \quad \text{for } m\in\mathbb{Z} \quad \text{and} \quad j=1,2,\dots,s, \tag{5.8}$$

where $\mathcal{T}_{U,a}\mathbf{F} = x$ and for $j=1,2,\dots,s$

$$\mathbf{g}_j(w) := \left(\sum_{k\in\mathbb{Z}} \mathcal{L}_j a_1(k)\,\mathrm{e}^{2\pi \mathrm{i} kw}, \sum_{k\in\mathbb{Z}} \mathcal{L}_j a_2(k)\,\mathrm{e}^{2\pi \mathrm{i} kw}, \dots, \sum_{k\in\mathbb{Z}} \mathcal{L}_j a_L(k)\,\mathrm{e}^{2\pi \mathrm{i} kw}\right)^\top$$

belongs to $L^2_L(0,1)$. As in the one-generator case we must study the sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rmw}\}_{m\in\mathbb{Z};j=1,2,\dots,s}$ in $L^2_L(0,1)$. Consider the $s\times rL$ matrix of functions in $L^2(0,1)$

$$\begin{aligned}\mathbb{G}(w) &:= \begin{bmatrix} \mathbf{g}_1^\top(w) & \mathbf{g}_1^\top\left(w+\frac{1}{r}\right) & \cdots & \mathbf{g}_1^\top\left(w+\frac{r-1}{r}\right) \\ \mathbf{g}_2^\top(w) & \mathbf{g}_2^\top\left(w+\frac{1}{r}\right) & \cdots & \mathbf{g}_2^\top\left(w+\frac{r-1}{r}\right) \\ \vdots & \vdots & & \vdots \\ \mathbf{g}_s^\top(w) & \mathbf{g}_s^\top\left(w+\frac{1}{r}\right) & \cdots & \mathbf{g}_s^\top\left(w+\frac{r-1}{r}\right) \end{bmatrix} \\ &= \left[\mathbf{g}_j^\top\left(w+\frac{k-1}{r}\right)\right]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r}} \end{aligned} \tag{5.9}$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)], \quad \beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)].$$

In [13, Lemma 2], one can find the proof of the following lemma.

Lemma 5.2. *Let $\mathbf{g}_j$ be in $L^2_L(0,1)$ for $j=1,2,\dots,s$ and let $\mathbb{G}(w)$ be its associated matrix given in (5.9). Then, the following results hold:*

(a) *The sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a complete system for $L^2_L(0,1)$ if and only if the rank of the matrix $\mathbb{G}(w)$ is rL a.e. in $(0,1/r)$.*
(b) *The sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Bessel sequence for $L^2_L(0,1)$ if and only if $\mathbf{g}_j \in L^\infty_L(0,1)$ (or equivalently $\beta_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{G}}/r$.*
(c) *The sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a frame for $L^2_L(0,1)$ if and only if $0<\alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.*
(d) *The sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z},j=1,2,\dots,s}$ is a Riesz basis for $L^2_L(0,1)$ if and only if is a frame and $s=rL$.*

In case that the sequence $\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};j=1,2,\dots,s}$ is a frame for $L^2_L(0,1)$ (here, necessarily $s\ge rL$), a dual frame is given by $\{r\mathbf{h}_j(w)\,\mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};j=1,2,\dots,s}$, where the functions $\mathbf{h}_j$, $j=1,2,\ldots,s$, form an $L\times s$ matrix $\mathbf{h}(w):=[\mathbf{h}_1(w), \mathbf{h}_2(w),\ldots,\mathbf{h}_s(w)]$ with entries in $L^\infty(0,1)$, and satisfying

$$[\mathbf{h}_1(w),\mathbf{h}_2(w),\ldots,\mathbf{h}_s(w)]\mathbb{G}(w)=[\mathbb{I}_L,\mathbb{O}_{L\times(r-1)L}] \quad \text{a.e. in } (0,1)$$

(see [13] for the details). That is, the matrix $\mathbf{h}(w)$ is formed with the first L rows of a left-inverse of the matrix $\mathbb{G}(w)$ having essentially bounded entries in $(0,1)$. In other words, all the dual frames of $\{\overline{\mathbf{g}_j(\mathrm{e}^{2\pi irnw})}\}_{n\in\mathbb{Z};j=1,2,\ldots,s}$ with the above property are obtained by taking the first L rows of the $rL\times s$ matrices given by

$$\mathbb{H}_{\mathbb{K}}(w):=\mathbb{G}^\dagger(w)+\mathbb{K}(w)[\mathbb{I}_s-\mathbb{G}(w)\mathbb{G}^\dagger(w)],$$

where $\mathbb{K}(w)$ denotes any $rL\times s$ matrix with entries in $L^\infty(0,1)$.

Thus, any $\mathbf{F}\in L^2_L(0,1)$ can be expanded as

$$\mathbf{F}=\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\langle\mathbf{F},\overline{\mathbf{g}_j(w)}\mathrm{e}^{2\pi irnw}\rangle_{L^2_L(0,1)}r\mathbf{h}_j(w)\mathrm{e}^{2\pi irnw} \quad \text{in } L^2_L(0,1).$$

Applying the isomorphism $\mathcal{T}_{U,a}$ and taken into account (5.8), for each $x=\mathcal{T}_{U,a}\mathbf{F}\in\mathcal{A}_{\mathbf{a}}$ we get the sampling expansion

$$x=\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\mathcal{L}_jx(rn)U^{rn}[\mathcal{T}_{U,a}(r\mathbf{h}_j)]=\sum_{j=1}^{s}\sum_{n\in\mathbb{Z}}\mathcal{L}_jx(rn)U^{rn}c_{j,\mathbf{h}} \quad \text{in } \mathcal{H},$$

where the sampling elements $c_{j,\mathbf{h}}=\mathcal{T}_{U,a}(r\mathbf{h}_j)\in\mathcal{A}_{\mathbf{a}}$, $j=1,2,\ldots,s$, and the sequence $\{U^{rn}c_{j,\mathbf{h}}\}_{n\in\mathbb{Z};j=1,2,\ldots,s}$ is a frame for $\mathcal{A}_{\mathbf{a}}$. Proceeding as in Sec. 3, it is straightforward to state and prove the corresponding results.

The time-jitter error sampling formulas

Under appropriate slight changes, the time-jitter error results in Sec. 4 still remain valid for the case of multiple generators. Namely, given an error sequence $\boldsymbol{\epsilon}:=\{\epsilon_{mj}\}_{m\in\mathbb{Z};j=1,2,\ldots,s}$, assume that the operator

$$D_{\boldsymbol{\epsilon}}:\ell^2_L(\mathbb{Z})\to\ell^2_s(\mathbb{Z}),$$
$$\mathbf{c}\mapsto D_{\boldsymbol{\epsilon}}\mathbf{c}:=(D_{\boldsymbol{\epsilon},1}\mathbf{c},\ldots,D_{\boldsymbol{\epsilon},s}\mathbf{c}),$$

is well-defined, where $\mathbf{c}:=(\{c^1_k\}_{k\in\mathbb{Z}},\{c^2_k\}_{k\in\mathbb{Z}},\ldots,\{c^L_k\}_{k\in\mathbb{Z}})\in\ell^2_L(\mathbb{Z})$ and, for $j=1,2,\ldots,s$,

$$D_{\boldsymbol{\epsilon},j}\mathbf{c}:=\left\{\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}[\mathcal{L}_ja_l(rm-k+\epsilon_{mj})-\mathcal{L}_ja_l(rm-k)]c^l_k\right\}_{m\in\mathbb{Z}}.$$

The operator norm (it could be infinity) is defined as usual

$$\|D_{\boldsymbol{\epsilon}}\|:=\sup_{\mathbf{c}\in\ell^2_L(\mathbb{Z})\setminus\{0\}}\frac{\|D_{\boldsymbol{\epsilon}}\mathbf{c}\|_{\ell^2_s(\mathbb{Z})}}{\|\mathbf{c}\|_{\ell^2_L(\mathbb{Z})}},$$

where $\|D_\epsilon\mathbf{c}\|^2_{\ell^2_s(\mathbb{Z})}:=\sum_{j=1}^{s}\|D_{\epsilon,j}\mathbf{c}\|^2_{\ell^2(\mathbb{Z})}$ and $\|\mathbf{c}\|^2_{\ell^2_L(\mathbb{Z})}=\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}|c^l_k|^2$ for each $\mathbf{c}\in\ell^2_L(\mathbb{Z})$. Assume that the matrix $\mathbb{G}$ in (5.9) satisfies $0<\alpha_{\mathbb{G}}\le\beta_{\mathbb{G}}<\infty$, and let $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};j=1,2,\ldots,s}$ be an error sequence satisfying the inequality $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$. Then, proceeding as in Sec. 4, there exists a frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};j=1,2,\ldots,s}$ for $\mathcal{A}_{\mathbf{a}}$ such that, for any $x \in \mathcal{A}_{\mathbf{a}}$ a sampling formula as in (4.2) holds.

Now assume that $b_j \in D_T$, $j = 1, 2, \ldots, s$; thus the functions $\mathcal{L}_{b_j} a_l(t) \equiv \mathcal{L}_j a_l(t)$, $j = 1, 2, \ldots, s$ and $l = 1, 2, \ldots, L$, are continuously differentiable on $\mathbb{R}$. Again, as in Sec. 4, under the decay condition (4.4) for each $(\mathcal{L}_j a_l)'(t)$, $j = 1, 2, \ldots, s$ and $l = 1, 2, \ldots, L$, one can easily prove that there exists $\delta > 0$ such that $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}| < \delta$ for each $j = 1, 2, \ldots, s$, implies that $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$ for the error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};j=1,2,\ldots,s}$.

Acknowledgments

The authors wish to thank the referees for their valuable and constructive comments. This work has been supported by the grant MTM2009–08345 from the Spanish *Ministerio de Ciencia e Innovación* (MICINN).

References

[1] B. Adcock and A. C. Hansen, Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon, *Appl. Comput. Harmon. Anal.* **32** (2012) 357–388.

[2] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Dover Publications, New York, 1993).

[3] A. Aldroubi, Oblique projections in atomic spaces, *Proc. Amer. Math. Soc.* **124**(7) (1996) 2051–2060.

[4] A. Aldroubi, Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces, *Appl. Comput. Harmon. Anal.* **13** (2002) 151–161.

[5] A. Aldroubi and K. Gröchenig, Non-uniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.* **43** (2001) 585–620.

[6] A. Aldroubi, Q. Sun and W. S. Tang, Convolution, average sampling and a Calderon resolution of the identity for shift-invariant spaces, *J. Fourier Anal. Appl.* **11**(2) (2005) 215–244.

[7] M. S. Birman and M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space* (Springer, New York, 1987).

[8] O. Christensen, *An Introduction to Frames and Riesz Bases* (Birkhäuser, Boston, 2003).

[9] O. Christensen and Y. C. Eldar, Oblique dual frames and shift-invariant spaces, *Appl. Comput. Harmon. Anal.* **17**(1) (2004) 48–68.

[10] O. Christensen and Y. C. Eldar, Generalized shift-invariant systems and frames for subspaces, *Appl. Comput. Harmon. Anal.* **11**(3) (2005) 299–313.

[11] I. Djokovic and P. P. Vaidyanathan, Generalized sampling theorems in multiresolution subspaces, *IEEE Trans. Signal Process.* **45** (1997) 583–599.

[12] H. R. Fernández-Morales, A. G. García and G. Pérez-Villalón, Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators, in *Multiscale Signal Analysis and Modeling*, Lecture Notes in Electrical Engineering (Springer, New York, 2012), pp. 51–80.

[13] A. G. García, M. A. Hernández-Medina and G. Pérez-Villalón, Generalized sampling in shift-invariant spaces with multiple stable generators, *J. Math. Anal. Appl.* **337** (2008) 69–84.

[14] A. G. García and G. Pérez-Villalón, Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces, *Appl. Comput. Harmon. Anal.* **20**(3) (2006) 422–433.
[15] A. G. García and G. Pérez-Villalón, Generalized irregular sampling in shift-invariant spaces, *Int. J. Wavelets Multiresolut. Inf. Process.* **5**(3) (2007) 369–387.
[16] A. G. García and G. Pérez-Villalón, Multivariate generalized sampling in shift-invariant spaces and its approximation properties, *J. Math. Anal. Appl.* **355** (2009) 397–413.
[17] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, 1999).
[18] S. Kang and K. H. Kwon, Generalized average sampling in shift-invariant spaces, *J. Math. Anal. Appl.* **377** (2011) 70–78.
[19] A. N. Kolmogorov, Stationary sequences in Hilbert space, *Boll. Moskow. Gos. Univ. Mat.* **2** (1941) 1–40.
[20] S. Li and H. Ogawa, Pseudo-duals of frames with applications, *Appl. Comput. Harmon. Anal.* **11** (2001) 289–304.
[21] S. Li and H. Ogawa, Pseudoframes for subspaces with applications, *J. Fourier Anal. Appl.* **10**(4) (2004) 409–431.
[22] T. Michaeli, V. Pohl and Y. C. Eldar, U-invariant sampling: Extrapolation and causal interpolation from generalized samples, *IEEE Trans. Signal Process.* **59**(5) (2011) 2085–2100.
[23] A. Papoulis, Generalized sampling expansion, *IEEE Trans. Circuits Syst.* **24** (1977) 652–654.
[24] V. Pohl and H. Boche, U-invariant sampling and reconstruction in atomic spaces with multiple generators, *IEEE Trans. Signal Process.* **60**(7) (2012) 3506–3519.
[25] H. L. Royden, *Real Analysis*, 3rd edn. (Macmillan Publishing Company, New York, 1988).
[26] M. H. Stone, On one-parameter unitary groups in Hilbert spaces, *Ann. Math.* **33**(3) (1932) 643–648.
[27] W. Sun and X. Zhou, Average sampling in shift-invariant subspaces with symmetric averaging functions, *J. Math. Anal. Appl.* **287** (2003) 279–295.
[28] M. Unser and A. Aldroubi, A general sampling theory for nonideal acquisition devices, *IEEE Trans. Signal Process.* **42**(11) (1994) 2915–2925.
[29] M. Unser and J. Zerubia, A generalized sampling theory without band-limiting constraints, *IEEE Trans. Circuits Syst. II* **45**(8) (1998) 959–969.
[30] R. Venkataramani and Y. Bresler, Sampling theorems for uniform and periodic nonuniform MIMO sampling of multiband signals, *IEEE Trans. Signal Process.* **51**(12) (2003) 3152–3163.
[31] G. G. Walter, A sampling theorem for wavelet subspaces, *IEEE Trans. Inform. Theory* **38** (1992) 881–884.
[32] J. Weidmann, *Linear Operators in Hilbert Spaces* (Springer, New York, 1980).
[33] K. Yosida, *Functional Analysis*, 6th edn. (Springer, New York, 1995).
[34] J. Zhou, Multidimensional multirate systems characterization, design and applications, thesis, University of Illinois (2005).